Now (1) will have either one or three real roots. If three, two of them by Descarte's Rule of Signs will be positive and one negative. In the former case the quantity $9/m^2-4/m$ under the radical sign in (2) will be positive. In the latter case it will be negative. Consequently the point of change from the reducible to the irreducible case of Cardan will be when $9/m^2-4/m=0$, which gives $1/m=\frac{4}{9}$. Substituting this value in (1) we have $r^3-Rr^2+4R^3/27=0$..(3), whose roots are $r_1=-R/3$, and $r_2=r_3=\frac{2}{3}R$. The positive roots being equal shows that $\frac{4}{9}$ is the maximum value of $1/m$ which will give positive roots.

II. Solution by the PROPOSER.

Let a be the radius of the hollow sphere, and x the diameter of the solid sphere, and, therefore, the weight of the water. Then the center of the hollow sphere is $x-a$ or $a-x$ from the surface of the water; and we find that $\sqrt{(2ax-x^2)}$=radius of surface of the water, and as the amount of water=one half the cylinder having area of surface of water for base and height of water for altitude it is equal to $\frac{1}{2}\pi x^2(2a-x)=(4\pi na^3)/3$, (using n instead of $1/m$). Reducing we have $x^3-2ax^2=(-8na^3)/3$. This being a cubic equation, it has three roots; and as their product is negative, one, or all of them must be negative; but as their sum is positive at least one of them must be positive; and taking both of them together, there must be two positive roots, and therefore x has two values, each of which answers the conditions. From our equation we have $n=3(2ax^2-x^3)/8a^3$. Differentiating and reducing we find $x=4a/3$, and substituting this in the expression for the value of n, we have $n=1/m=\frac{4}{9}$. When $1/m=\frac{4}{9}$, $x=4a/3$, and the equation has two equal positive roots—*practically* but not *mathematically*, an exception to the proposition that x has two values, each of which satisfies the conditions.

Also solved by *H. C. WHITAKER* and *G. B. M. ZERR*.

121. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(x^5+y^5+z^5)^3+(x+y)^2=31$.

$(x^5+y^5+z^5)^3+(x+y+z)^3=729$.

$(x+y)^2+(x+y+z)^3=31$.

Solution by W. F. BUCK, Instructor in the Science Department, Leominster High School, Leominster, Mass.

Let $x^5+y^5+z^5=r$....(4).

$x+y=s$....(5).

$x+y+z=t$....(6).

Then from the original equations,

$$r^3+s^2=31, \quad r^3+t^3=729, \quad s^2+t^3=31,$$

which easily give

$$r=\sqrt[3]{\tfrac{729}{2}}, \quad s=\sqrt{-\tfrac{667}{2}}, \quad t=\sqrt[3]{\tfrac{729}{2}}\ldots.(7).$$

From (5) and (6), $z=t-s$. Therefore, from (4), $x^5+y^5=r-(t-s)^5$.

Dividing this by (5) and subtracting (5) raised to fourth power from the result

$$x^2+xy+y^2=\frac{r-(t-s)^5-s^5}{-5sxy}.$$

But from (5), $x^2+xy+y^2=s^2-xy$.

$$\therefore\ x^2y^2-s^2xy=\frac{r-(t-s)^5-s^5}{5s},\ \text{and}\ xy=\tfrac{1}{2}s^2\pm\sqrt{\frac{r-(t-s)^5-s^5}{5s}+\frac{s^4}{4}}\ldots(8).$$

Subtracting (8) multiplied by 4 from (5) squared, and taking square root obtain $x-y$, from which with (5),

$$x=\tfrac{1}{2}s\pm\tfrac{1}{2}\sqrt{-s^2\mp4\sqrt{\frac{r-(t-s)^5}{5s}+\frac{s^4}{4}}}$$

whence x by substitution from (7), then x and y from (5) and (6).

Also solved by *J. SCHEFFER*, and *G. B. M. ZERR*.

122. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A man buys a five per cent. ten-year bond at such a price as enables him to spend annually three per cent. upon his investment and by continually investing the residue of the annual interest and its increase annually at four per cent., at the end of term upon payment of his bond has his original investment. What price per \$100 does he pay for the bond?

Solution by D. G. DORRANCE, JR., Camden, Oneida County, N. Y.; and H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.

At the beginning of the time, the man pays $\$x$ for the bond; at the end of the time, he receives \$100 and an accumulated annuity of ($\$5-.03x$) running for 10 years at 4%. The value of the annuity at that time was $25(1.04^{10}-1)(5-.03x),=12.0061(5-.03x)$.

Hence $x=100+12.0061(5-.03x)$, from which $x=\$117.6537$.

By a slightly different construction which allows for only *nine* years' expenditure, Professor Zerr obtains the result \$116.548.

MECHANICS.

118. Proposed by M. E. ANDERSON, Minneapolis, Minn.

A closed steel cylinder of length L and diameter D is placed in a horizontal position. The cylinder is filled with water to a depth (a) from the lower side, the space above the water being filled with air at a pressure P_1.

What work will be done against this increasing pressure, and against gravity, by a pump forcing water into this tank until the pressure has increased to P_2? Suppose the level of the water in the tank at the beginning to be the same as that of the reservoir from which the water is pumped.